13. Double integrals over Rectangles, Iterated Integrals

In this lecture, we will discuss

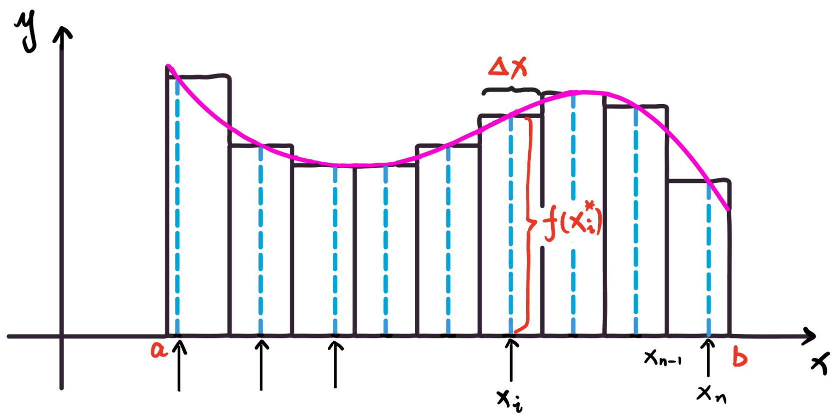
- Review of the Definite Integral
- Double Integrals over Rectangles
  - Defintion of the double integral of f over the rectangle R
  - Midpoint Rule for Double Integrals
  - Average Value
  - Properties of Double Integrals
- Iterated Integrals
  - Definition of iterated integrals
  - Fubini's Theorem moved to Lecture 14

Review of the Definite Integral

Riemann sum of functions of a single variable

Let $f(x)$ be a function defined for $a \leqslant x \leqslant b$, we start by dividing the interval $[a, b]$ into n subintervals $[x_{i-1}, x_i]$ of equal width $\Delta x = \dfrac{b-a}{n}$ and we choose sample points x_i^* in these subintervals. Then we form the Riemann sum

$$\sum_{i=1}^{n} f\left(x_i^*\right) \Delta x$$



Then we take the limit of such sums as $n \rightarrow \infty$ to obtain the definite integral of f from a to b:

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^{n} f\left(x_i^*\right) \Delta x$$

In the special case where $f(x) \geqslant 0$, the Riemann sum can be interpreted as the sum of the areas of the approximating rectangles in the above figure ,

and $\displaystyle\int_a^b f(x) dx$ represents the area under the curve $y = f(x)$ from a to b.

Double Integrals over Rectangles

Volumes and Double Integrals

Now we consider a function f of two variables defined on a closed rectangle

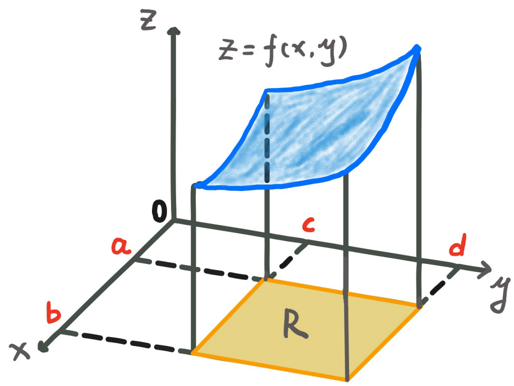
$$R = [a, b] \times [c, d] = \left\{(x, y) \in \mathbb{R}^2 \mid a \leqslant x \leqslant b, c \leqslant y \leqslant d\right\}$$

and for simplicity we first suppose that $f(x, y) \geqslant 0$.

The graph of f is a surface with equation $z = f(x, y)$. Let S be the solid that lies above R and under the graph of f, that is,

$$S = \left\{(x, y, z) \in \mathbb{R}^3 \mid 0 \leqslant z \leqslant f(x, y), (x, y) \in R\right\}$$

We want to find the volume of S.



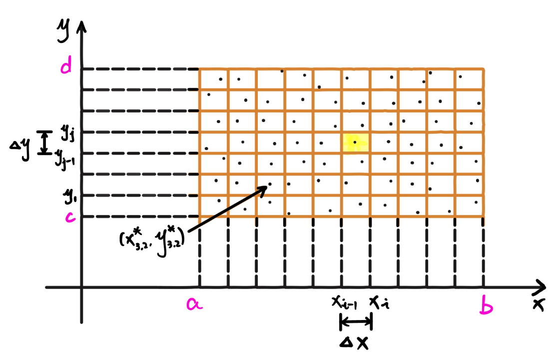
We start from dividing the rectangle R into subrectangles.

We do this by dividing the interval $[a, b]$ into m subintervals $[x_{i-1}, x_i]$ of equal width $\Delta x = \dfrac{b-a}{m}$ and dividing $[c, d]$ into n subintervals $[y_{j-1}, y_j]$ of equal width $\Delta y = \dfrac{d-c}{n}$.

This allows us to form the subrectangles

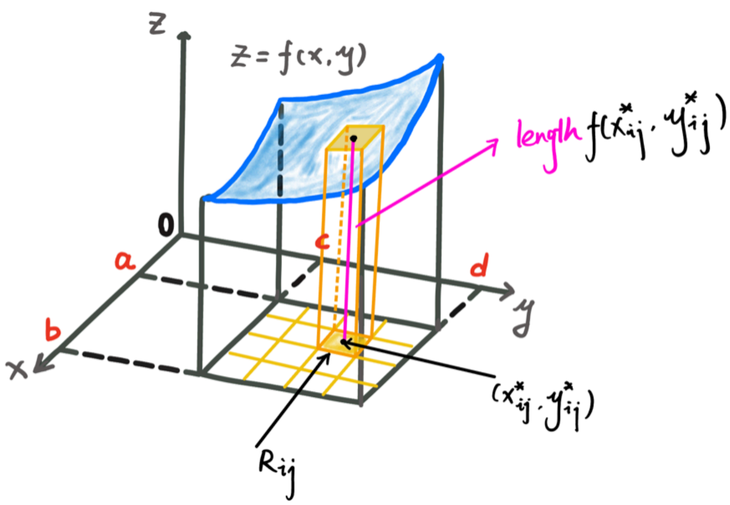
$$R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] = \{(x, y) \mid x_{i-1} \leqslant x \leqslant x_i, y_{j-1} \leqslant y \leqslant y_j\}$$

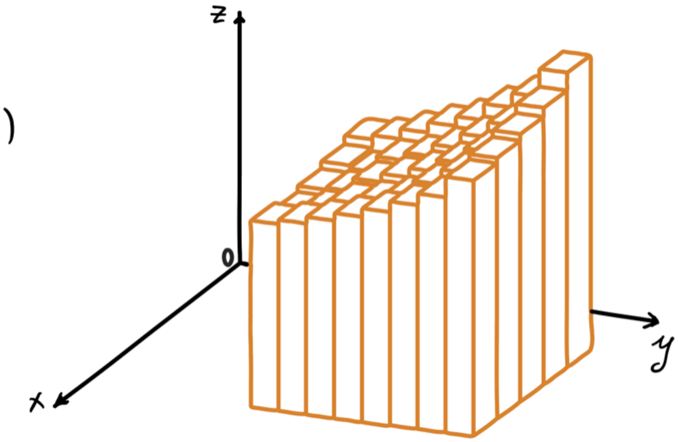
each with area $\Delta A = \Delta x \Delta y$.



If we choose a sample point $\left(x_{ij}^{*}, y_{ij}^{*}\right)$ in each R_{ij}, then we can approximate the part of S that lies above each R_{ij} by a thin rectangular box with base R_{ij}

and height $f\left(x_{ij}^{*}, y_{ij}^{*}\right)$ as the left figure below.





The volume of this box is the height of the box times the area of the base rectangle:

$$f\left(x_{ij}^{*}, y_{ij}^{*}\right) \Delta A$$

We follow this process for all the rectangles and add the volumes of the corresponding boxes, we get an approximation to the total volume of S :

$$V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f\left(x_{ij}^{*}, y_{ij}^{*}\right) \Delta A$$

This double sum means that for each subrectangle we evaluate f at the chosen point and multiply by the area of the subrectangle, and then we add the results. See the right figure above.

The approximation of the volumn V becomes better as m and n become larger and so we would expect that

$$V = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f\left(x_{ij}^{*}, y_{ij}^{*}\right) \Delta A$$

We use this expression of V to define the volume of the solid S that lies under the graph of f and above the rectangle R.

The sample point $\left(x_{ij}^{*}, y_{ij}^{*}\right)$ can be chosen to be any point in the subrectangle R_{ij}, but if we choose it to be the upper right-hand corner of R_{ij} , then the

expression for the double integral looks simpler.

Definition The double integral of f over the rectangle R is

$$\iint_R f(x, y) dA=\lim _{m, n \rightarrow \infty} \sum_{i=1}^m \sum_{j=1}^n f\left(x_i, y_j\right) \Delta A$$

if this limit exists.

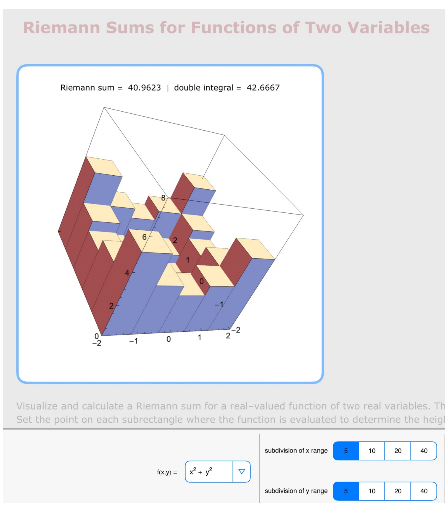
Remarks.

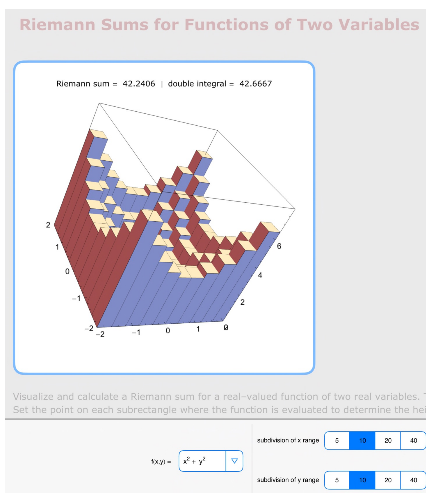
1. If $f(x, y) \geqslant 0$, then the volume V of the solid that lies above the rectangle R and below the surface $z=f(x, y)$ is

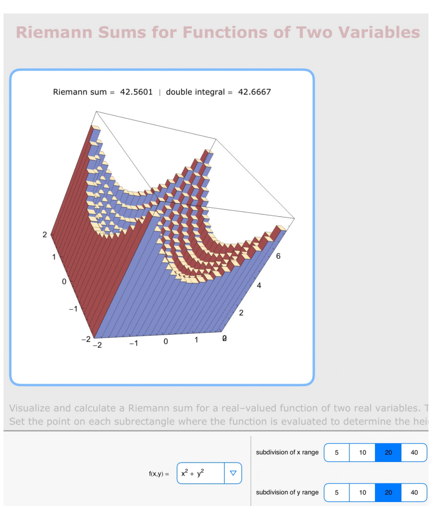
$$V=\iint_R f(x, y) dA$$

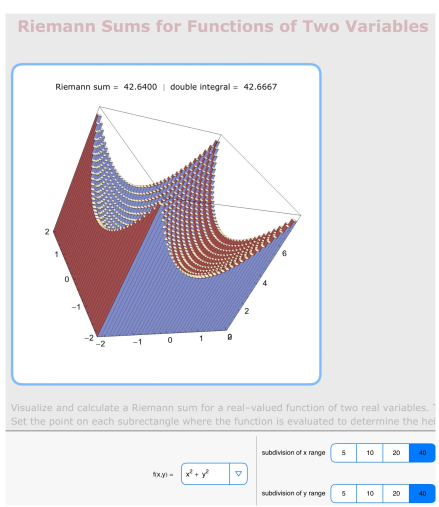
2. $\displaystyle\sum_{i=1}^m \sum_{j=1}^n f\left(x_{ij}^*, y_{ij}^*\right) \Delta A$ is called a double Riemann sum and is used as an approximation to the value of the double integral.

Example 0 Use the online WOLFRAM Demonstrations Project about [Riemann Sums for Functions of Two Variables]() to visualize the Riemann sums when we take different subintervals.









The Midpoint Rule

Recall in Calculus for single variable functions, we study the Midpoint Rule:

Review: The Midpoint Rule for Single Variable Function

Assume that $f(x)$ is continuous on $[a, b]$. If $[a, b]$ is divided into n subintervals, then

$$\int_a^b f(x)dx \approx \sum_{i=1}^{n} f(\bar{x}_i)\Delta x$$

where $\bar{x}_i$ is the midpoint of the i^{th} subinterval.

This can be generalized to the case of double integrals.

Theorem. Midpoint Rule for Double Integrals

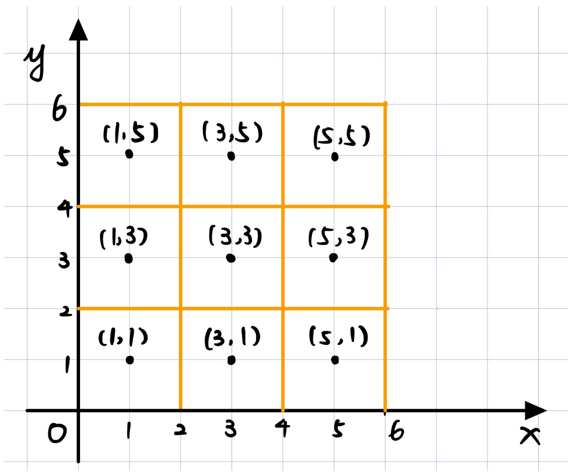
$$\iint_R f(x, y)dA \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(\bar{x}_i, \bar{y}_j)\Delta A$$

where $\bar{x}_i$ is the midpoint of $[x_{i-1}, x_i]$ and $\bar{y}_j$ is the midpoint of $[y_{j-1}, y_j]$.

Example 1.

Let $R = [0,6] \times [0,6]$. Subdivide each side of R into $m = n = 3$ subintervals, and use the Midpoint Rule to estimate the value of

$$\iint_R (2y - x^2)\,dA$$



Let $f(x,y) = 2y - x^2$.

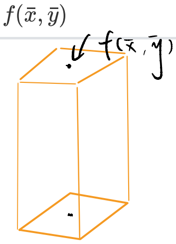
We divide $R = [0,6] \times [0,6]$ into 3×3 sub-rectangles.

Since $m = n = 3$.

We know $\Delta A = 2 \times 2 = 4$ is the area of each sub-rectangle.

We compute the values of $f(\bar{x}, \bar{y})$ for each of the above 9 rectangles (squares).

Midpoints	$(1,1)$	$(1,3)$	$(1,5)$	$(3,1)$	$(3,3)$	$(3,5)$	$(5,1)$	$(5,3)$	$(5,5)$
$f(\bar{x}, \bar{y})$	1	5	9	-7	-3	1	-23	-19	-15



E.g. when $(\bar{x}, \bar{y}) = (1,1)$, $f(1,1) = 2 \cdot 1 - 1^2 = 1$

$f(1,3) = 2 \cdot 3 - 1^2 = 5$

Therefore. $\iint_R (2y - x^2)\,dA$

$$= \sum_{i=1}^{3} \sum_{j=1}^{3} f(\bar{x}_i, \bar{y}_i)\,\Delta A$$

$$= 4\,(1 + 5 + 9 - 7 - 3 + 1 - 23 - 19 - 15)$$

$$= 4 \cdot (-51)$$

$$= -204$$

Average Value

Recall the average value of a function f of one variable defined on an interval $[a, b]$ is

$$f_{\text{ave}} = \frac{1}{b-a} \int_a^b f(x) dx$$

Similarly, we define the average value of a function f of two variables defined on a rectangle R to be

$$f_{\text{ave}} = \frac{1}{A(R)} \iint_R f(x, y) dA$$

where $A(R)$ is the area of R.

If $f(x, y) \geqslant 0$, we have

$$A(R) \times f_{\text{ave}} = \iint_R f(x, y) dA,$$

which states that the box with base R and height f_{ave} has the same volume as the solid that lies under the graph of f.

We will discuss an application of this after we introduce the iterated integrals.

Properties of Double Integrals

One can genearlize some properties of the integrals of single variable functions to double integral:

$$\iint_R [f(x, y) + g(x, y)] dA = \iint_R f(x, y) dA + \iint_R g(x, y) dA$$
$$\iint_R cf(x, y) dA = c \iint_R f(x, y) dA \quad \text{where } c \text{ is a constant}$$

If $f(x, y) \geqslant g(x, y)$ for all (x, y) in R, then

$$\iint_R f(x, y) dA \geqslant \iint_R g(x, y) dA$$

Iterated Integrals

In the following, we will discuss how to express a double integral as an iterated integral, which can then be evaluated by calculating two single integrals.

- Suppose that f is a function of two variables that is continuous on the rectangle $R = [a, b] \times [c, d]$.
- We use the notation $\int_c^d f(x, y) dy$ to mean that x is held fixed and $f(x, y)$ is integrated with respect to y from $y = c$ to $y = d$.
- This procedure is called partial integration with respect to y.
- Now $\int_c^d f(x, y) dy$ is a number that depends on the value of x, so it defines a function of x :

$$A(x) = \int_c^d f(x, y) dy$$

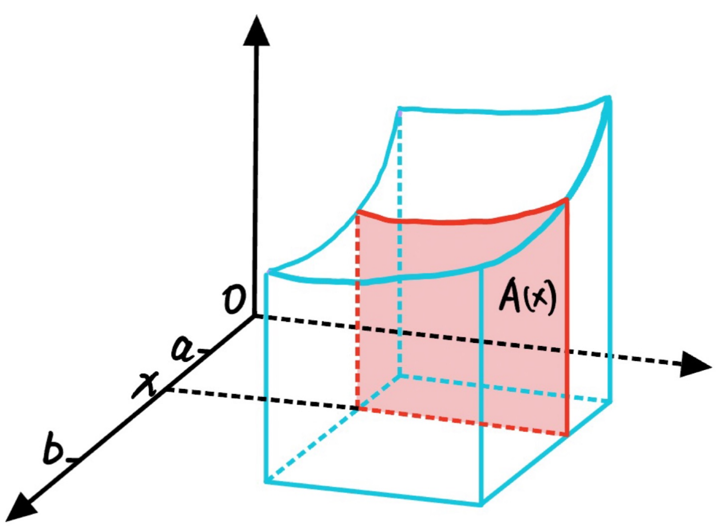
- If we now integrate the function A with respect to x from $x = a$ to $x = b$, we get

$$\int_a^b A(x) dx = \int_a^b \left[\int_c^d f(x, y) dy \right] dx$$

- Thus

$$\int_a^b \int_c^d f(x, y) dy dx = \int_a^b \left[\int_c^d f(x, y) dy \right] dx \tag{1}$$

The integral on the right side of Equation (1) is called an iterated integral, means that we first integrate with respect to y from c to d and then with respect to x from a to b.



$$A(x) = \int_c^d f(x, y)\, dy$$

$$Vol = \int_a^b A(x)\, dx$$

Example 2. Evaluate $\int_0^3 \int_0^\pi r^4 \sin\theta \, d\theta \, dr$

ANS: $\int_0^3 \left[\int_0^\pi r^4 \sin\theta \, d\theta \right] dr$ ← We begin with computing the inner integral in terms of θ.

We treat r as a scalar for this step.

$$= \int_0^3 r^4 \left[-\cos\theta \right] \Big|_0^\pi \, dr$$

$$= \int_0^3 r^4 \left[-\left(\underbrace{\cancel{\cos \pi}}_{-1} - \underbrace{\cancel{\cos 0}}_{1} \right) \right] dr$$

$$= \int_0^3 2r^4 \, dr$$

$$= \frac{2}{4+1} r^5 \Big|_0^3$$

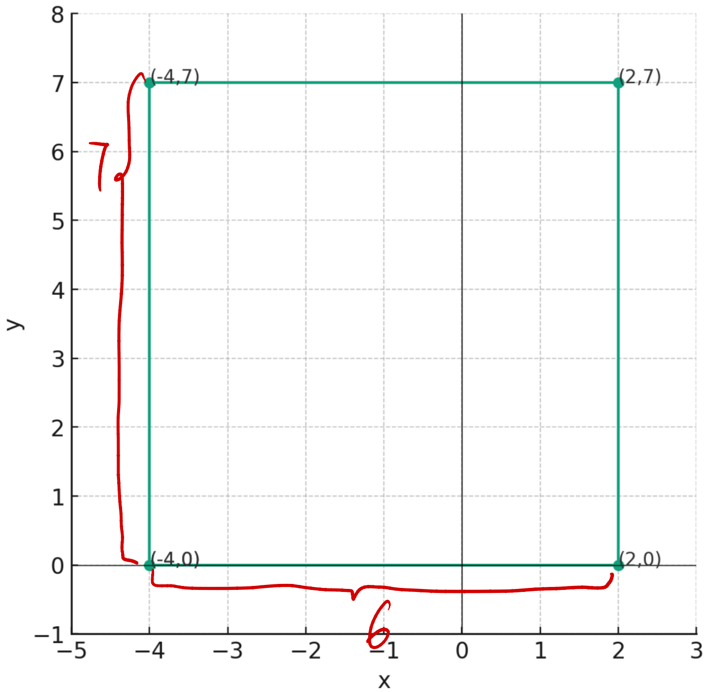
$$= \frac{2}{5} r^5 \Big|_0^3$$

$$= \frac{2}{5} \left(3^5 - 0^5 \right)$$

$$= \frac{2 \cdot 243}{5}$$

$$= \frac{486}{5}$$

Example 3. Find the average value of $f(x,y) = x^2y$ over the rectangle R with vertices $(-4,0), (-4,7), (2,7), (2,0)$



ANS: Recall the average value of a function $f(x,y)$ on a rectangle R is

$$f_{ave} = \frac{1}{A(R)} \iint_R f(x,y)\, dA$$

Note the area of R: $A(R) = 6 \times 7 = 42$.

We also need to compute

$$\iint_R f(x,y)\, dA = \int_{-4}^{2} \int_0^7 x^2 y \, dy\, dx$$

$$= \int_{-4}^{2} \frac{1}{2} x^2 y^2 \Big|_0^7 dx = \int_{-4}^{2} \frac{1}{2} x^2 \left[7^2 - 0^2\right] dx$$

$$= \int_{-4}^{2} \frac{49}{2} x^2\, dx = \frac{49}{2} \cdot \frac{1}{3} x^3 \Big|_{-4}^{2}$$

$$= \frac{49}{2} \cdot \frac{1}{3} \left[2^3 - (-4)^3\right] = \frac{49}{2} \cdot \frac{1}{3} (8 + 64) = 49 \times 12$$

Thus $f_{ave} = \dfrac{1}{A(R)} \iint_R f(x,y)\, dA = \dfrac{49 \times 12}{42} = 14$.

Exercise 4. The following is a collection of some further exercises for the interate integrals

(1) Evaluate the iterated integral:

$$\int_0^2 \int_6^8 \sqrt{x+4y}\, dx\, dy \qquad \text{Recall } \int x^a\, dx = \frac{1}{1+a} x^{1+a} + C$$

ANS: $\int_0^2 \int_6^8 \sqrt{x+4y}\; dx\, dy = \int_0^2 \int_6^8 (x+4y)^{\frac{1}{2}} dx\, dy$

$= \int_0^2 \frac{1}{\frac{1}{2}+1} (x+4y)^{\frac{1}{2}+1} \Big|_6^8 dy = \int_0^2 \frac{2}{3} (x+4y)^{\frac{3}{2}} \Big|_6^8 dy$

$= \frac{2}{3} \int_0^2 \left[(8+4y)^{\frac{3}{2}} - (6+4y)^{\frac{3}{2}} \right] dy = \frac{2}{3} \left[\int_0^2 (8+4y)^{\frac{3}{2}} dy - \int_0^2 (6+4y)^{\frac{3}{2}} dy \right]$

$= \frac{2}{3} \left[\frac{1}{4} \int_0^2 (8+4y)^{\frac{3}{2}} d(8+4y) - \frac{1}{4} \int_0^2 (6+4y)^{\frac{3}{2}} d(6+4y) \right]$

$= \frac{2}{3} \left[\frac{1}{4} \cdot \overset{\frac{2}{5}}{\cancel{\frac{1}{1+\frac{3}{2}}}} (8+4y)^{1+\frac{3}{2}} \Big|_0^2 - \frac{1}{4} \int_0^2 \overset{\frac{2}{5}}{\cancel{\frac{1}{1+\frac{3}{2}}}} (6+4y)^{1+\frac{3}{2}} \Big|_0^2 \right]$

$= \frac{2}{3} \left[\frac{1}{10} \left[(8+8)^{\frac{5}{2}} - 8^{\frac{5}{2}} \right] - \frac{1}{10} \left[(6+8)^{\frac{5}{2}} - 6^{\frac{5}{2}} \right] \right]$

$= \frac{1}{15} \left(16^{5/2} - 8^{5/2} - 14^{5/2} + 6^{5/2} \right) \approx 13.1865$

(2) Calculate $\int_0^{\frac{3\pi}{2}} \int_0^{\pi} (x\sin(y) - y\cos(x))\, dx\, dy$

Note by the property of double integral, we can rewrite the given integral as

$$\underbrace{\int_0^{\frac{3\pi}{2}} \int_0^{\pi} x\sin y \, dx\, dy}_{①} - \underbrace{\int_0^{\frac{3\pi}{2}} \int_0^{\pi} y\cos x \, dx\, dy}_{②}$$

Then we compute ① & ② separately

$$① \Rightarrow \int_0^{\frac{3\pi}{2}} \int_0^{\pi} x\sin y \, dx\, dy = \int_0^{\frac{3\pi}{2}} \sin y \left(\tfrac{1}{2}x^2\right)\Big|_0^{\pi} dy$$

$$= \int_0^{\frac{3\pi}{2}} \frac{\pi^2}{2} \sin y \, dy = \frac{\pi^2}{2}\left[-\cos y\right]\Big|_0^{\frac{3\pi}{2}} = \frac{\pi^2}{2}\left[-\cancel{\cos\tfrac{3\pi}{2}} + \cancel{\cos 0}\right]$$

1

$$= \frac{\pi^2}{2}$$

$$② \Rightarrow \int_0^{\frac{3\pi}{2}} \int_0^{\pi} y\cos x \, dx\, dy = \int_0^{\frac{3\pi}{2}} y \sin x \Big|_0^{\pi} dy$$

$$= \int_0^{\frac{3\pi}{2}} 0 \, dy = 0$$

Therefore, the given integral is

$$\underbrace{\int_0^{\frac{3\pi}{2}} \int_0^{\pi} x\sin y \, dx\, dy}_{①} - \underbrace{\int_0^{\frac{3\pi}{2}} \int_0^{\pi} y\cos x \, dx\, dy}_{②}$$

$$= \frac{\pi^2}{2} - 0$$

$$= \frac{\pi^2}{2}$$

(3) Evaluate the integral $\int_0^{\pi/4} \int_0^1 (y\cos x + 1)\, dy\, dx$.

ANS: $\int_0^{\frac{\pi}{4}} \int_0^1 (y \cos x + 1)\, dy\, dx$

$= \int_0^{\frac{\pi}{4}} \left[\frac{1}{2} y^2 \cos x + y \right] \Big|_0^1 dx$

$= \int_0^{\frac{\pi}{4}} \left(\frac{1}{2} \cos x + 1 \right) dx$

$= \left(\frac{1}{2} \sin x + x \right) \Big|_0^{\frac{\pi}{4}}$

$= \frac{1}{2} \sin \frac{\pi}{4} + \frac{\pi}{4}$

$= \frac{\sqrt{2}}{4} + \frac{\pi}{4}$

(3) Evaluate the integral $\int_0^{\pi/4} \int_0^1 (y\cos x + 1)\, dy\, dx$.

ANS: $\int_0^{\frac{\pi}{4}} \int_0^1 (y \cos x + 1)\, dy\, dx$

$= \int_0^{\frac{\pi}{4}} \left[\frac{1}{2} y^2 \cos x + y\right]\Big|_0^1 dx$

$= \int_0^{\frac{\pi}{4}} \left(\frac{1}{2}\cos x + 1\right) dx$

$= \left(\frac{1}{2}\sin x + x\right)\Big|_0^{\frac{\pi}{4}}$

$= \frac{1}{2}\sin\frac{\pi}{4} + \frac{\pi}{4}$

$= \frac{\sqrt{2}}{4} + \frac{\pi}{4}$